Adaptive pole-placement of controllable systems*

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Abstract For controllable systems, a computationally simplified adaptive pole-placement control is introduced, which leads the input and output of the closed-loop system to be bounded. Compared with previous work, modifications to parameter estimates may stop at specially designed stopping times and the number of modifications decreases from infinite to finite. Further, the dimension of parameter to be modified is reduced so that the computational load is gradually lessened.

Keywords: adaptive control, parameter estimation, pole-placement, Sylvester matrix.

A necessary and sufficient condition for linear systems with known coefficients to be arbitrarily pole-placed is that the system is controllable. It is desirable to solve the adaptive pole-placement problem also under the controllability condition.

For minimum-phase systems, the boundedness of system output guarantees the boundedness of system input. There have been many works where adaptive controls are given so that the system input and output are bounded in the closed-loop, for instance, references [1, 2].

For non-minimum-phase systems, in order to get a stable closed-loop system, various methods are adopted to construct adaptive controls, for instance, the self-excitation method of Kreisselmeier[3] and Giri[4], the large-excitation method of Chen and Zhang[5], the parameter modification method of Lozano, etc.[6] The system Kreisselmeier considered is with constant parameters and of disturbance-free and uncertainty-dynamics-free. The adaptive controls given by Kreisselmeier have self-excitation ability, i.e. when the estimated parameters are not satisfactory so that the system input or output diverges, then the control law will use the divergent input or output signals to force the estimated parameters to approach their true values, and further, make the performance of the closed-loop system better. Giri *et al*. apply this idea to solving the adaptive control problems for systems with bounded disturbances, unmodeled dynamics and slowly time-varying parameters[4, 7]. Chen and Zhang investigate stochastic systems with constant parameters and average bounded disturbances. They introduce external signal to the involved system to improve the signal-noise ratio, so as to get more exact parameter estimates, and then, design adaptive control via certainty equivalence principle[5]. For systems with constant parameters, bounded

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disturbances and unmodeled dynamics, Lozano *et al*. combine dead-beat method with the least-squares method to estimate system parameters, and show that the estimated parameters are self-convergent. They use this self-convergent property to modify the estimated parameters to overcome the difficulty stemming from singularity of the Sylvester matrix corresponding to the estimated parameters, and theoretically, present a pole-placement-based adaptive stabilization method[6]. Recently, for stochastic systems with constant parameters and average bounded disturbances, Guo[8], based on the self-convergence of the weighted least-squares, gives a stochastic search method which can improve the controllability of the closed-loop systems and needs only to compute at most two determinants at each time.

This paper introduces a stochastic stopping time method such that the estimated parameters are modified not at each time and the modification will stop in a finite time. Based on the rank of the limiting matrix of $F(t)$, the dimension of the parameters to be modified is adaptively reduced, and the computational load is gradually lessened. For example, for a 3-dimensional system, if the rank of the limiting matrix of $F(t)$ is 1, then according to the methods of this paper, only 7 determinants of 6-dimensional matrices and 1 singular value decomposition need to be computed in contrast to 46 656 determinants of the same dimensional matrices as required in ref. [6]. Further, only a finite number of modifications is needed in this paper, while in ref. [6] the modification should be continued forever. By using the stopping time method of this paper, the input and output of the closed-loop system are bounded under the computationally simplified adaptive pole-placement control if the open-loop system is controllable.

1 System model and parameter estimation

Suppose that the system is described by the following difference equation:

$$A(z)y(t)=B(z)u(t)+w(t), \tag{1.1}$$

where $u(t)$, $y(t)$, $w(t)$ are system input, output and disturbances, respectively; z is the unit shift-back operator,

$$A(z)=1+a_1z+\cdots+a_nz^n,\ B(z)=b_1z+\cdots+b_nz^n. \tag{1.2}$$

Denote the unknown coefficients in $A(z)$ and $B(z)$ by $\theta\overset{\Delta}{=}[a_1,\ \cdots,\ a_n,\ b_1,\ \cdots,\ b_n]^{\tau}$, and set

$$\varphi(t-1)^{\tau}=[-y(t-1),\ \cdots,\ -y(t-n),\ u(t-1),\ \cdots,\ u(t-n)]. \tag{1.3}$$

Then (1.1) can be expressed as

$$y(t)=\theta^{\tau}\varphi(t-1)+w(t). \tag{1.4}$$

For different disturbances $w(t)$, different parameter estimation algorithms will be adopted. Precisely, the following two cases will be dealt with, respectively.

Case 1.1. $w(t)$ is the dynamics uncertainty, but there are two known nonnegative constants η and μ such that $|w(t)| \leqslant \eta + \mu \|\varphi(t-1)\|$. In this situation, the dead-beat least-squares method is introduced to estimate the unknown parameter θ. Let

$$\overline{\varphi}(t-1) = \frac{\varphi(t-1)}{1+\|\varphi(t-1)\|}, \quad \overline{y}(t) = \frac{y(t)}{1+\|\varphi(t-1)\|}, \tag{1.5}$$

$$\delta(t) = \mu + \frac{\eta}{1+\|\varphi(t-1)\|}, \quad \mathrm{e}(t) = \overline{y}(t) - \theta^{\tau}(t-1)\overline{\varphi}(t-1), \tag{1.6}$$

$$v(t) = \mathrm{e}(t)^2 + \overline{\varphi}^{\tau}(t-1)F^2(t-1)\overline{\varphi}(t-1), \tag{1.7}$$

$$\overline{\delta}(t) = (1+\mathrm{tr}F(0))(\delta(t)^2 + \varepsilon\delta(t)), \tag{1.8}$$

$$\lambda(t) = \begin{cases} 0, & \text{if } v(t) \leqslant \overline{\delta}(t), \\ 1, & \text{if } v(t) > \overline{\delta}(t), \end{cases} \tag{1.9}$$

where $\varepsilon > 0$ is a given positive constant; $\mathrm{tr}F(0)$ denotes the trace of $F(0)$.

The dead-beat least-squares method can recursively be presented as

$$\theta(t) = \theta(t-1) + \lambda(t) F(t) \overline{\varphi}(t-1)\mathrm{e}(t), \tag{1.10}$$

$$F(t) = F(t-1) - \frac{\lambda(t-1)F(t-1)\overline{\varphi}(t-1)\overline{\varphi}^{\tau}(t-1)F(t-1)}{1+\lambda(t)\overline{\varphi}^{\tau}(t-1)F(t-1)\overline{\varphi}(t-1)}, \tag{1.11}$$

where initial values $\theta(0)$ and $F(0) > 0$ are arbitrarily chosen.

Case 1.2. $\{w(t), \mathscr{F}_t\}$ is a martingale difference sequence, and

$$\sup_{t \geqslant 0} E[w^2(t+1) | \mathscr{F}_t] < \infty \quad \text{a.s.}$$

In this situation, the weighted least-squares method is used to estimate θ:

$$\theta(t+1) = \theta(t) + \frac{P(t)\varphi(t)}{f(t) + \varphi^{\tau}(t)P(t)\varphi(t)}[y(t+1) - \theta^{\tau}(t)\varphi(t)], \tag{1.12}$$

$$P(t+1) = P(t) - \frac{P(t)\varphi(t)\varphi^{\tau}(t)P(t)}{f(t) + \varphi^{\tau}(t)P(t)\varphi(t)}, \tag{1.13}$$

where initial values $\theta(0)$ and $P(0) > 0$ are arbitrarily chosen,

$$f(t) = \log\left(1 + \|P(0)\| + \sum_{i=0}^{t} \|\varphi(i)\|^2\right)^{1+\delta}, \quad \delta > 0. \tag{1.14}$$

In the case where $A(z)$ and $B(z)$ are coprime, refs. [6] and [8] show that $\theta(t)$, $F(t)$ and $P(t)$ given by (1.5)—(1.11) and (1.12)—(1.14) are self-convergent, i.e. for any adapted control $\{u(t)\}$, $\theta(t)$, $F(t)$ and $P(t)$ converge to finite limits as t goes to ∞.

2 Adaptive control based on finite number of parameter modification

When the adaptive pole-placement control is designed, according to the certainty equivalence principle the first problem one has to deal with is the possible degeneracy of the Sylvester matrix resulting from the estimated parameter $\theta(t)$. To overcome this difficulty, ref. [6] introduces a vector $\beta(t)$ to modify $\theta(t)$, and uses the modified value

$$\overline{\theta}(t)=\theta(t)+F(t)\beta(t) \tag{2.1}$$

to replace $\theta(t)$ in the Sylvester matrix, where for Case 1.1, $F(t)$ is given by (1.11), while for Cases 1.2, $F(t)=P^{1/2}(t)$.

$\beta(t)$ introduced in ref. [6] has the following properties.

Property 2.1. *$\beta(t)$ is bounded.*

Property 2.2. *$\beta(t)$ makes the absolute value of the determinant of Sylvester matrix $M(\overline{\theta}(t))$ corresponding to $\overline{\theta}(t)$ uniformly greater than a positive constant for a $t_0\geqslant 0$ and all $t\geqslant t_0$.*

Express $\overline{\theta}(t)$ in blocks:

$$\overline{\theta}(t)=[\overline{a}_1(t),\ \cdots,\ \overline{a}_n(t),\ \overline{b}_1(t),\ \cdots,\ \overline{b}_n(t)]^{\tau}, \tag{2.2}$$

and set

$$\overline{A}(t,\ z)=1+\overline{a}_1(t)z+\cdots+\overline{a}_n(t)z^n,\ \overline{B}(t,\ z)=\overline{b}_1(t)z+\cdots+\overline{b}_n(t)z^n. \tag{2.3}$$

Then $M(\overline{\theta}(t))$ is non-singular, where

$$M(\overline{\theta}(t))\triangleq\begin{pmatrix} 1 & \overline{a}_1(t)\cdots\cdots & \cdots & \cdots\overline{a}_n(t) & 0 & \cdots & 0 \\ 0 & 1 & \ddots & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 & \overline{a}_1(t) & \cdots\cdots & \overline{a}_n(t) \\ 0 & \overline{b}_1(t)\cdots\cdots & \cdots & \cdots\overline{b}_n(t) & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \overline{b}_1(t) & \cdots\cdots & \overline{b}_n(t) \end{pmatrix}^{\tau} \tag{2.4}$$

Therefore, $\overline{A}(t,\ z)$ and $\overline{B}(t,\ z)$ are coprime. Furthermore, for any stable polynomial $C(z)$ with order less than or equal to $2n-1$, polynomials $R(t,\ z)$ and $S(t,\ z)$ are solvable from

$$\overline{A}(t,\ z)S(t,\ z)+\overline{B}(t,\ z)R(t,\ z)=C(z). \tag{2.5}$$

Suppose that $y^*(t)$ is a known, bounded signal. According to the certainty equivalence principle, the pole-placement adaptive control should be

$$u(t)=[1-S(t,\ z)]u(t)-R(t,\ z)y(t)+C(z)y^*(t). \tag{2.6}$$

From refs. [6, 8] we have the following

Proposition 2.1. *Suppose that* $A(z)$ *and* $B(z)$ *are coprime, and that* $\beta(t)$ *in* (2.1) *has Properties* 2.1 *and* 2.2.

Case 1.1. If $\theta(t)$ and $F(t)$ are generated by (1.5)—(1.11), then there is $\bar{\mu}>0$ such that for any $\mu\in[0, \bar{\mu})$, the input and output of the closed-loop system are bounded under the adaptive pole-placement control given by (2.1)—(2.6)[6].

Case 1.2. If the parameter estimates are given by the weighted least-squares method (1.12)—(1.14), then the input and output of the closed-loop system are bounded under the adaptive pole-placement control given by (2.1)—(2.6)[8].

Parameter $\beta(t)$ in (2.1) is modified in the following way in reference [6]:

Let $m=2n$, $l=m^{m-1}$, and take arbitrarily ml positive numbers $\sigma_1, \sigma_2, \cdots, \sigma_{ml}$ such that $\sigma_i\geqslant\sigma_{i-1}+1$. Set $\beta_i=[\sigma_i, \sigma_i^m, \cdots, \sigma_i^l]^\tau$, $\mathscr{M}=\{\beta_i, i=1, 2, \cdots, ml\}$, and define

$$h_t(\beta(t))\triangleq|\det M(\theta(t)+F(t)\beta(t))|, \tag{2.7}$$

$$\beta(t)=\begin{cases}\beta(t-1), & \text{if } h_t(\beta_i)<(1+\gamma)h_t(\beta(t-1)),\ \forall\beta_i\in\mathscr{M},\\ \beta_j, & j=\min\{i: h_t(\beta_i)\geqslant(1+\gamma)h_t(\beta(t-1)), \text{ and } h_t(\beta_i)\geqslant h_t(\beta_s),\ \forall\beta_s\in\mathscr{M}\},\end{cases}$$

where γ is a small enough positive number.

From this, we see that such a parameter modification requires computing $(2n)^{2n}$ determinants of $2n$-dimensional matrices at each step, and the modification procedure will not stop at any finite time.

In this section, we introduce a stopping time, and make the modification finished in finite steps. In the next section, we will reduce the dimension of $\beta(t)$, and hence, gradually reduce the computational load.

Let $\tau_1=1$. Select $\beta_{\tau_k}\in\mathscr{M}$ such that

$$h_{\tau_k}(\beta_{\tau_k})=\max_{\beta_i\in\mathscr{M}} h_{\tau_k}(\beta_i). \tag{2.8}$$

Define

$$\tau_{k+1}=\inf\left\{t: t>\tau_k,\ h_t(\beta_{\tau_k})\leqslant\frac{1}{2}h_{\tau_k}(\beta_{\tau_k})\right\}, \tag{2.9}$$

$$\beta(t)=\beta_{\tau_k},\ t\in[\tau_k, \tau_{k+1}). \tag{2.10}$$

Theorem 2.1. *Suppose that* $A(z)$ *and* $B(z)$ *are coprime. Let* $\theta(t)$ *and* $F(t)$ *be generated by* (1.5)—(1.11) *for Case* 1.1, *and let* $\theta(t)$ *and* $P(t)$ *be generated by* (1.12)—(1.14) *for Case* 1.2 *with* $F(t)=P^{1/2}(t)$. *If* $\beta(t)$ *is chosen by* (2.9) *and* (2.10), *then under the adaptive control given by* (2.1)—(2.6), *there exists an integer* k_0 *such that* $\tau_{k_0}=\infty$; *that is,* $\beta(t)$ *will take a constant value after finite steps of modification. In this situation, for Case* 1.1 *there* *is* $\bar{\mu}>0$ *such that for any* $\mu\in[0, \bar{\mu})$, *the input and output of the closed-loop system are bounded, and for Cases* 1.2, *the input and output of the closed-loop system are bounded.*

Proof. It suffices to prove that there exists an integer k_0 such that $\tau_{k_0}=\infty$. In this case, according to (2.10), $\beta(t)=\beta_{\tau_{k_0}}$ for all $t\geqslant\tau_{k_0}$, and Properties 2.1 and 2.2 are satisfied by $\beta(t)$. Then, the boundedness of the system input and output follows from Proposition 2.1.

Since $\theta(t)$ and $F(t)$ are convergent,

$$h_t(\beta_i)\triangleq|\det M(\theta(t)+F(t)\beta_i)|\xrightarrow[t\to\infty]{}h(\beta_i)\geqslant 0,\ \text{for all}\ \beta_i\in\mathscr{M}.$$

Divide $\mathscr{M}$ into $\mathscr{M}=\mathscr{M}_1\cup\mathscr{M}_2$ such that $h(\beta_i)>0$ for any β_i in $\mathscr{M}_1$, and $h(\beta_i)=0$ for any β_i in $\mathscr{M}_2$.

From eq. (4.33) of ref. [6] it follows that $\sum_{i=1}^{ml}h(\beta_i)>0$. Thus $\mathscr{M}_1$ is a nonempty subset of $\mathscr{M}$. Let $h\triangleq\min_{\beta_i\in\mathscr{M}_1}h(\beta_i)$. Then $h>0$, since $\mathscr{M}$ is finite and $\mathscr{M}_1$ is a nonempty subset of $\mathscr{M}$.

In the sequel, we will frequently use the following simple fact: if $f(t)\xrightarrow[t\to\infty]{}f>0$, then there exists T such that

$$f(t)\geqslant\frac{1}{2}f(s)>0,\quad \text{for all}\ t,\ s\geqslant T. \tag{2.11}$$

For any $\beta_j\in\mathscr{M}_2$, there exists $T'(\beta_j)$ such that $h_t(\beta_j)<2h/3$ for all $t\geqslant T'(\beta_j)$. Thus,

$$|\det M(\theta(t)+F(t)\beta_i)|>\frac{2}{3}h>|\det M(\theta(t)+F(t)\beta_j)|\quad \forall\beta_i\in\mathscr{M}_1\ \text{and}\ \beta_j\in\mathscr{M}_2, \tag{2.12}$$

if $t\geqslant\max_{\beta_i\in\mathscr{M}_1,\ \beta_j\in\mathscr{M}_2}\{T(\beta_i),\ T'(\beta_j)\}$. This implies that β_{τ_k} selected according to (2.8) must belong to $\mathscr{M}_1$ for large enough t. Therefore, $\beta_{\tau_k}\in\mathscr{M}_1$ if k is large enough (i.e. τ_k large enough). Further, from (2.11) it follows that

$$h_t(\beta_{\tau_k})>\frac{1}{2}h_s(\beta_{\tau_k}),\ \forall t\geqslant T(\beta_{\tau_k}),\ \forall s\geqslant T(\beta_{\tau_k}). \tag{2.13}$$

Let $T=\max_{\beta_i\in\mathscr{M}_1}T(\beta_i)$. It is clear that $T<\infty$, since $\mathscr{M}_1$ is a finite set. If k is large enough, then $\tau_k>T$, and it follows from (2.13) that $h_t(\beta_{\tau_k})>\frac{1}{2}h_{\tau_k}(\beta_{\tau_k})$, $\forall t\geqslant\tau_k>T(\beta_{\tau_k})$. Thus, there exists an integer k_0 such that $\tau_{k_0}=\infty$.

3 Dimension reduction for parameters under modification

In the adaptive control law designed in the last section, a stopping time is adopted such that parameter $\beta(t)$ is no longer to be modified after finite steps. This saves a lot of computational load. However, since $\beta(t)$ is $2n$-dimensional, the method used above requires computing $(2n)^{2n}$ determinants of $2n$-dimensional matrices at each step, and this is very time-consuming. In this section, we try to reduce the dimension of $\beta(t)$ so as to essentially lessen the computational load. Taking a 3-dimensional system as an example, 46 656 determinants of 6-dimensional matrices need to be computed at each step. However, if the dimension of $\beta(t)$ can be reduced by 2, i.e. from 6 to 4, then the number of determinants to be computed at each step will be reduced to 1 296, and the computational load will be decreased by more than 97%.

Lemma 3.1. *Suppose that* $F(t) \xrightarrow[t\to\infty]{} F$, *the rank of* F *is* j, *and that there is an orthogonal matrix* $U(t)$ *such that*

$$U(t)F(t)U^{\tau}(t)=\mathrm{Diag}(\lambda_1(t),\ \cdots,\ \lambda_{2n}(t))\ \textit{and}\ \lambda_1(t)\geqslant\lambda_2(t)\geqslant\cdots\geqslant\lambda_{2n}(t).$$

For integers $i\leqslant k\leqslant 2n$, $U_{i:k}(t)$ *denotes the matrix consisting of the ith to kth rows of* $U(t)$ *in their original order. For integers* $s\leqslant 2n$ *and* $i=1,\ \cdots,\ (2n)^s$, *set* $\beta_{i,\ s}=[\sigma_i,\ \sigma_i^{2n},\ \cdots,\ \sigma_i^{(2n)^{s-1}}]^{\tau}$, *where* $\sigma_i\geqslant\sigma_{i-1}+1$.

Suppose further that θ *satisfies* $\det M(\theta)\neq 0$. *Then there is a constant* $\varepsilon>0$ *independent of* t *and* $F(t)$ *such that for any large enough* t,

$$\sup_{i=1,\ 2,\ \cdots,\ (2n)^s}|\det M(\theta(t)+U_{1:s}^{\tau}(t)\beta_{i,\ s})|>\varepsilon,\ \forall s\in\{j,\ j+1,\ \cdots,\ 2n\}.$$

Proof of Lemma 3.1. Let $\beta^*(t)=F^{-1}(t)(\theta-\theta(t))$. Then it follows from ref. [6] that $\{\beta_t^*\}$ is bounded. By a straightforward computation, we have

$$\begin{aligned}F(t)\beta^*(t)=&U_{1:s}^{\tau}(t)\mathrm{Diag}(\lambda_1(t),\ \cdots,\ \lambda_s(t))U_{1:s}(t)\beta^*(t)\\&+U_{(s+1):(2n)}^{\tau}(t)\mathrm{Diag}(\lambda_{s+1}(t),\ \cdots,\ \lambda_{2n}(t))U_{(s+1):(2n)}(t)\beta^*(t).\end{aligned}$$

Noticing that the second term on the right-hand side of the above equation converges to zero, and $\det M(\theta(t)+F(t)\beta^*(t))=\det M(\theta)\neq 0$, we have

$$\det M(\theta(t)+U_{1:s}^{\tau}(t)v(t))\neq 0,\ \text{for all large enough}\ t,$$

where $v(t)=\mathrm{Diag}(\lambda_1(t),\ \cdots,\ \lambda_s(t))U_{1:s}(t)\beta^*(t)$.

Replace F_t in Theorem 2 of ref. [6] by $U_{1:s}^{\tau}(t)$, and let $v(\sigma_i)=[1,\ \sigma_i,\ \cdots,\ \sigma_i^{m^m-1}]^{\tau}$. Then (4.21)—(4.25) of ref. [6] still hold, and θ^* and β_t^* in (4.26) are replaced by $\theta(t)+U_{1:s}^{\tau}(t)v(t)$ and $v(t)$, respectively. Then the remaining part of the proof of that theorem is still true. This implies the desired result.

Lemma 3.2. *Under the conditions of Lemma* 3.1, *there is a bounded* $\{\beta(t)\}$ *such that*

$$U_{1:s}^{\tau}(t)\beta_{i,\ s}=F(t)\beta(t),\ \textit{for any}\ \beta_{i,\ s},\ s\leqslant j.$$

Proof. Take $\beta(t)=U_{1:s}^{\tau}(t)\mathrm{Diag}(\lambda_1^{-1}(t),\ \cdots,\ \lambda_s^{-1}(t))$. It is evident that $\{\beta(t)\}$ is bounded. From

$$F(t)=[U_{1:s}^{\tau}(t),\ U_{(s+1):(2n)}^{\tau}(t)]\mathrm{Diag}(\lambda_1(t),\ \cdots,\ \lambda_{2n}(t))\begin{bmatrix}U_{1:s}(t)\\U_{(s+1):(2n)}(t)\end{bmatrix},$$

it follows that $F(t)\beta(t)=U^{\tau}_{1:s}(t)\beta_{i,\,s}$.

Let $j(t)$ denote the rows of $U_{1:j(t)}(t)$, and write $U(t)$ in blocks: $U(t)=[U^{\tau}_{1:j(t)}(t), U^{\tau}_{(j(t)+1):(2n)}(t)]^{\tau}$. Specifically, in the case where $j(t)=0$ we agree that $U_{1:j(t)}(t)=0$.

Take initial values $\tau_1=0$, $j(\tau_1)=0$ and $\varepsilon_{\tau_1}\in\left(0,\ \frac{1}{2}\lambda_{2n}(\tau_1)\right)$. For $k=1$ (and further, recursively for $k=2, 3, \cdots$) define $\beta_{k(\tau_i),\,j(\tau_i)}$ such that

$$\begin{aligned}&|\det M(\theta(\tau_i)+U^{\tau}_{1:j(\tau_i)}(\tau_i)\beta_{k(\tau_i),\,j(\tau_i)})|\\ =&\max_{k=1,\,2,\,\cdots,\,(2n)^{j}(\tau_i)}|\det M(\theta(\tau_i)+U^{\tau}_{1:j(\tau_i)}(\tau_i)\beta_{k,\,j(\tau_i)})|\overset{\Delta}{=}g_{\tau_i}(\beta_{k(\tau_i),\,j(\tau_i)}),\end{aligned}\tag{3.1}$$

where $g_t(\beta_{k,\,j(\tau_i)})=|\det M(\theta(t)+U^{\tau}_{1:j(\tau_i)}(t)\beta_{k,\,j(\tau_i)})|$.

Define

$$\tau'_{i+1}=\inf\{t:\ t>\tau_i \text{ and } \lambda_{j(\tau_i)}(t)\leqslant\varepsilon_{\tau_i}\},\tag{3.2}$$

$$\tau''_{i+1}=\inf\left\{t:\ t>\tau_i \text{ and } g_t(\beta_{k(\tau_i),\,j(\tau_i)})\leqslant\frac{1}{2}g_{\tau_i}(\beta_{k(\tau_i),\,j(\tau_i)})\right\},\tag{3.3}$$

$$\tau'''_{i+1}=\begin{cases}\tau_i+1, & \text{if } g_{\tau_i}(\beta_{k(\tau_i),\,j(\tau_i)})=0,\\ \infty, & \text{otherwise;}\end{cases}\tag{3.4}$$

$$\tau_{i+1}=\min\{\tau'_{i+1},\ \tau''_{i+1},\ \tau'''_{i+1}\},\tag{3.5}$$

$$j(\tau_{i+1})=j(\tau_i)-1,\ \text{if } \tau'_{i+1}<\tau''_{i+1} \text{ and } \tau'''_{i+1}=\infty,\tag{3.6}$$

$$j(\tau_{i+1})=j(\tau_i)+1,\ \text{if } \tau'''_{i+1}<\infty,\ \text{or } \tau'_{i+1}\geqslant\tau''_{i+1} \text{ and } \tau'''_{i+1}=\infty;\tag{3.7}$$

$$\varepsilon_{j(\tau_{i+1})}=\frac{1}{2}\lambda_{j(\tau_{i+1})}(\tau_{i+1}).\tag{3.8}$$

Lemma 3.3. *Suppose that $F(t)\underset{t\to\infty}{\longrightarrow}F$ and the rank of F is j. Then there exists an integer i_0 such that $\tau_{i_0+1}=\infty$, and $j(\tau_{i_0})\leqslant j$, i.e. the dimension of $\beta_{k(\tau_{i_0}),\,j(\tau_{i_0})}$ is not greater than j.*

Proof. Denote by S the totality of $\beta_{i,\,k}$ where i runs over $\{1, 2, \cdots, (2n)^k\}$ and k runs over $\{1, 2, \cdots, 2n\}$. S is obviously bounded. Since $\theta(t)$ and $U(t)$ converge, we have

$$g_t(\beta_{i,\,k})\overset{\Delta}{=}|\det M(\theta(t)+U^{\tau}_{1:k}(t)\beta_{i,\,k})|\underset{t\to\infty}{\longrightarrow}g(\beta_{i,\,k})\geqslant 0,\ \forall\beta_{i,\,k}\in S.$$

For any $g(\beta_{i,\,k})>0$, according to (2.11) there exists $T(\beta_{i,\,k})$ such that

$$g_t(\beta_{i,\,k})>\frac{1}{2}g_s(\beta_{i,\,k}),\ \forall t,\ s\geqslant T(\beta_{i,\,k}).\tag{3.9}$$

Separate S into S_1 and S_2 such that $S=S_1\cup S_2$ and

$$\beta_{i,\,k}\in\begin{cases}S_1, & \text{if } g(\beta_{i,\,k})>0,\\ S_2, & \text{if } g(\beta_{i,\,k})=0.\end{cases}$$

Since $\mathscr{M}_1$ in Theorem 2.1 is a subset of S_1, S_1 is a nonempty subset of S. Set $g=\min_{\beta_{i,k}\in S_1} g(\beta_{i,k})$. Then $g>0$ because S is finite and S_1 is a subset of S.

For each $\beta_{p,q}\in S_2$, there is $T'(\beta_{p,q})$ such that

$$g_t(\beta_{p,q})=|\det M(\theta(t)+U^{\tau}_{1:q}(t)\beta_{p,q})|$$
$$<\frac{2}{3}g<g_t(\beta_{i,k}),\ \forall t\geqslant \max_{\beta_{i,k}\in S_1}\{T(\beta_{i,k}),\ T'(\beta_{p,q})\}. \tag{3.10}$$

Noticing that $F(t)\xrightarrow[t\to\infty]{}F$ and the rank of F is j, we have $\lambda_k(t)\xrightarrow[t\to\infty]{}\lambda_k>0$ for $k\leqslant j$. Thus from (2.11) it follows that, for large enough s,

$$\lambda_k(t)>\frac{1}{2}\lambda_k(s),\quad \forall t\geqslant s. \tag{3.11}$$

Suppose the converse, i.e. $\tau_i<\infty$ for all i. Since the dimension of $F(t)$ is finite, (3.6) occurs finitely many times. This is because, otherwise, (3.7) occurs infinitely many times, and this is impossible.

It follows from (3.11) that for large enough i,

$$\lambda_{j(\tau_i)}(t)>\frac{1}{2}\lambda_{j(\tau_i)}(\tau_i)=\varepsilon_{\tau_i},\ \text{if } j(\tau_i)\leqslant j. \tag{3.12}$$

Hence, for large enough i, if (3.6) occurs, then from (3.2) and (3.12) we must have $j(\tau_i)>j$. In this case, according to Lemma 3.1, there must be a $j(\tau_i)$-dimensional vector $\beta_{p,j(\tau_i)}\in S_1$. Therefore, from (3.10) it is known that $\beta_{k(\tau_i),j(\tau_i)}$ given by (3.1) belongs to S_1 whenever (3.6) occurs, if i is large enough. Recalling (3.9) we know that when i is large enough, if (3.6) occurs, then (3.7) is impossible for $s>i$. Thus, if (3.6) occurs infinitely often, then (3.6) is only possible for all large enough i. However, once (3.6) occurs, $j(\tau_i)$ will reduce by 1, and hence, after finitely many steps there must be $j(\tau_i)\leqslant j$. This, however, is impossible according to (3.12). Therefore, it is impossible that (3.6) occurs infinitely often. Hence, there exists i_0 such that $\tau_{i_0+1}=\infty$.

We now suppose that $j(\tau_{i_0})>j$. Then $\lambda_{j(\tau_{i_0})}(t)\xrightarrow[t\to\infty]{}0$. From (3.2) it follows that $\tau'_{i_0+1}<\infty$, so $\tau_{i_0+1}<\infty$. This contradicts $\tau_{i_0+1}=\infty$. Thus, $j(\tau_{i_0})\leqslant j$.

Define $\overline{\theta}(t)=\theta(t)+U^{\tau}_{1:j(\tau_i)}(t)\beta_{k(\tau_i),j(\tau_i)}$ for $t\in[\tau_i,\ \tau_{i+1})$ as the modification of $\theta(t)$.

Theorem 3.1. *Suppose that* $A(z)$ *and* $B(z)$ *are coprime, and that* $\theta(t)$ *and* $F(t)$ *are generated by* (1.5)—(1.11) *for Case* 1.1, *while* $\theta(t)$ *and* $P(t)$ *are given by* (1.12)—(1.14) *for Case* 1.2 *with* $F(t)=P^{1/2}(t)$. *Define the adaptive pole-placement control*:

$$u(t)=\begin{cases}[1-S(t,\ z)]u(t)-R(t,\ z)y(t)+C(z)y^*(t), & \textit{if } \det M(\overline{\theta}(t))\neq 0,\\ 0, & \textit{otherwise},\end{cases}$$

where $S(t, z)$, $R(t, z)$ and $M(\overline{\theta}(t))$ are generated by (2.1)—(2.5); $C(z)$ and $y^*(t)$ are the same as those in the last section. Then in Case 1.1, there is a $\overline{\mu}>0$ such that the input and output of the closed-loop system are bounded for any $\mu\in[0, \overline{\mu})$; while in Case 1.2, the input and output of the closed-loop system are bounded.

Proof. From Lemma 3.3 we have

$$|\det M(\theta(t)+U^{\tau}_{1:j(\tau_{i_0})}(t)\beta_{k(\tau_{i_0}),\ j(\tau_{i_0})})|>\frac{1}{2}g_{\tau_{i_0}}(\beta_{k(\tau_{i_0}),\ j(\tau_{i_0})})>0,\ \text{for}\ t\geqslant\tau_{i_0}.$$

Since $j(\tau_{i_0})\leqslant j$, by Lemma 3.2 we see that there is a bounded $\beta(t)$ such that

$$\overline{\theta}(t)=\theta(t)+U^{\tau}_{1:j(\tau_{i_0})}(t)\beta_{k(\tau_{i_0}),\ j(\tau_{i_0})}=\theta(t)+F(t)\beta(t),\ \text{for}\ t\geqslant\tau_{i_0}.$$

Thus, the modification here not only has the same structure as in (2.1), but also makes $\{\beta(t)\}$ satisfy Properties 2.1 and 2.2. The theorem follows directly from Proposition 2.1.

Remark. In Case 1.2, similar to ref. [8], the adaptive LQG problem can be solved by introducing an attenuating excitation signal, provided that $A(z)$ and $B(z)$ are coprime. It is worth noticing that here, $\beta(t)$ is modified only for finitely many times.

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